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Regularity of refinable functions with exponentially decaying masks ☆

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ABSTRACT

The regularity of refinable functions is an important issue in all multiresolution analysis and has a strong impact on applications of wavelets to image processing, geometric and numerical solutions of elliptic partial differential equations. The purpose of this paper is to characterize the regularity of refinable functions with exponentially decaying masks and a dilation matrix whose eigenvalues have the same modulus. The main results of this paper are really extensions of some results in Cohen et al. (1999) [5], Jia (1999) [17] and Lorentz and Oswald (2000) [28].

1. Introduction

A homogeneous refinement equation with mask a and a general dilation matrix M is the functional equation of the form

$$\phi(x) = \sum_{\alpha\in\mathbb{Z}^s} a(\alpha)\phi(Mx-\alpha), \quad x \in \mathbb{R}^s, \tag{1.1}$$

where ϕ is the unknown function defined on the s-dimensional Euclidean space $\mathbb{R}^s$, a is an exponentially decaying sequence on $\mathbb{Z}^s$ called a refinement mask, and M is an $s\times s$ integer matrix with $m = |\det M|$ such that $\lim_{n\to\infty} M^{-n} = 0$. The solution of (1.1) is called a refinable function, and the matrix M is called a dilation matrix. It is well known that refinement equations play an important role in wavelet analysis and computer graphics. Most useful wavelets in applications are generated from refinable functions.

Great efforts have been spent on the study of refinable functions when mask a is a finitely supported sequence (see, e.g., [1,5,6,9,13,17,29,32] and many references therein). However, in some cases, one need to study the refinement equations with infinitely supported masks. For example, in engineering, infinitely supported masks are needed [12]. The masks of various types of fractional splines in [36] are infinitely supported. In particular, due to some desirable properties, infinitely supported masks with exponential decay are of interest in the area of digital signal processing in electrical engineering (see e.g., [2,3,7,15,31]). To study Riesz bases of wavelets generated from multiresolution analysis, Han and Jia [14] investigated the L_2-solution of refinement equation (1.1) with an exponentially decaying mask and a general dilation matrix. Li and Yang [27] studied the existence of L_2-solution of vector refinement equation with an exponentially decaying mask and a general dilation matrix. In the binary case ($s=1$, $M=(2)$) and mask a is an exponentially decaying sequence, Cohen and Daubechies [3] studied the regularity of refinable functions by analyzing the spectral properties of transfer operators. Han [12] also investigated the regularity of refinable functions. In the case $s>1$ and $M = 2I_{s\times s}$, Lorentz and Oswald [28]

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investigated the regularity of refinable functions for the study of hierarchical bases in Sobolev spaces. In the case $s > 1$ and M is a general dilation matrix, Han and Jia [14] provided a sufficient condition to characterize the regularity of refinable functions with an exponentially decaying mask and a general dilation matrix.

Our goal is to investigate the regularity of solutions of refinement equation (1.1) by studying the spectral properties of associated transfer operators, and a formula for the critical Sobolev exponent is obtained, which extends some main results in [5,17] and [28] to the general setting. Our characterizations were inspired by the work of Cohen and Daubechies [3], Jia [17], Cohen, Gröchenig, and Villemoes [5] and Lorentz and Oswald [28].

Before proceeding further, we introduce some notations. Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{N}_0$ denote the set of nonnegative integers. An element $\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{N}_0^s$ is called a multi-index. For $j = 1, \ldots, s$, let e_j be the jth coordinate unit vector in $\mathbb{R}^s$. We denote by $\ell(\mathbb{Z}^s)$ the linear space of all (complex-valued) sequences on $\mathbb{Z}^s$, and by $\ell_0(\mathbb{Z}^s)$ the linear space of all finitely supported sequences on $\mathbb{Z}^s$. The difference operator ∇_j on $\ell(\mathbb{Z}^s)$ is defined by $\nabla_j a := a - a(\cdot - e_j)$, $a \in \ell(\mathbb{Z}^s)$. For $\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{N}_0^s$, ∇^μ is the difference operator $\nabla_1^{\mu_1} \cdots \nabla_s^{\mu_s}$, and D^μ is the differential operator $D_1^{\mu_1} \cdots D_s^{\mu_s}$.

As usual, for $1 \leqslant p \leqslant \infty$, we denote by $L_p(\mathbb{R}^s)$ the Banach space of all (complex-valued) functions f such that $\|f\|_p < \infty$, where

$$\|f\|_p := \left(\int_{\mathbb{R}^s} \left|f(x)\right|^p dx \right)^{1/p} \quad \text{for } 1 \leqslant p < \infty,$$

and $\|f\|_\infty$ is the essential supremum of f on $\mathbb{R}^s$. We observe that $L_2(\mathbb{R}^s)$ is a Hilbert space with the inner product given by

$$\langle f, g \rangle := \int_{\mathbb{R}^s} f(x)\overline{g(x)}\, dx.$$

Analogously, let $\ell_p(\mathbb{Z}^s)$ $(1 \leqslant p \leqslant \infty)$ be the Banach space of all (complex-valued) sequences $a = (a(\alpha))_{\alpha \in \mathbb{Z}^s}$ such that $\|a\|_p < \infty$, where

$$\|a\|_p := \left(\sum_{\alpha \in \mathbb{Z}^s} \left|a(\alpha)\right|^p \right)^{1/p} \quad \text{for } 1 \leqslant p < \infty,$$

and $\|a\|_\infty$ is the supremum of a on $\mathbb{Z}^s$.

Let b and c be two sequences on $\mathbb{Z}^s$, the convolution of b with c is defined by

$$b * c(\alpha) = \sum_{\beta \in \mathbb{Z}^s} b(\beta)c(\alpha - \beta), \quad \alpha \in \mathbb{Z}^s,$$

such that the sum is convergent. By the discrete version of Young's inequality, if $b \in \ell_1$ and $c \in \ell_p$ $(1 \leqslant p \leqslant \infty)$, then $b * c \in \ell_p$, and

$$\|b * c\|_p \leqslant \|b\|_1 \|c\|_p. \tag{1.2}$$

The Fourier transform of a function f in $L_1(\mathbb{R}^s)$ is defined by

$$\hat{f}(\xi) := \int_{\mathbb{R}^s} f(x) e^{-ix\cdot\xi}\, dx, \quad \xi \in \mathbb{R}^s,$$

where $\xi \cdot x$ is the inner product of two vectors ξ and x in $\mathbb{R}^s$. The domain of the Fourier transform can be naturally extended to functions in $L_2(\mathbb{R}^s)$ and tempered distribution. Similarly, the Fourier series of a sequence $c \in \ell_1(\mathbb{Z}^s)$ is defined by

$$\hat{c}(\xi) := \sum_{\alpha \in \mathbb{Z}^s} c(\alpha) e^{-i\alpha\cdot\xi}, \quad \xi \in \mathbb{R}^s.$$

Evidently, $\hat{c}$ is a 2π-periodic continuous function on $\mathbb{R}^s$. In particular, $\hat{c}$ is a trigonometric polynomial if c is finitely supported. We call $\hat{c}$, the symbol of c.

In order to clarify the concept of exponential decay, we introduce the weighted space E_μ as follows. Suppose c is a (complex-valued) summable sequence on $\mathbb{Z}^s$. For $\mu \geqslant 0$, define

$$\|\hat{c}\|_\mu := \sum_{\alpha \in \mathbb{Z}^s} \left|c(\alpha)\right| e^{\mu|\alpha|}. \tag{1.3}$$

Let E_μ denote the Banach space of all 2π-periodic functions $\hat{c}$ on $\mathbb{R}^s$ such that $\|\hat{c}\|_\mu < \infty$.

Weighted L_p spaces are defined as follows. Suppose f is a (complex-valued) measurable function on $\mathbb{R}^s$. For $\mu \geqslant 0$ and $1 \leqslant p < \infty$, define

$$\|f\|_{p,\mu} := \left(\int_{\mathbb{R}^s} \left| f(x) e^{\mu|x|} \right|^p dx \right)^{1/p}. \tag{1.4}$$

For $\mu \geqslant 0$, let $\|f\|_{\infty,\mu}$ be the essential supremum of the function $|f(x)|e^{\mu|x|}$ on $\mathbb{R}^s$. We use $L_{p,\mu}$ to denote the Banach space of all measurable functions f such that $\|f\|_{p,\mu} < \infty$.

In this paper, we always assume that the mask a is an exponentially decaying sequence, i.e., there exists a $\mu > 0$ such that $\hat{a} \in E_\mu$. Denote continuous function

$$H(\xi) := \frac{1}{m} \sum_{\alpha \in \mathbb{Z}^s} a(\alpha) e^{-i\alpha \cdot \xi}, \quad \xi \in \mathbb{R}^s. \tag{1.5}$$

To investigate the regularity of a refinable function associated with the exponentially decaying mask a and a dilation matrix M, following [3,18,28], we shall also assume that

$$H(\xi) = p(\xi) q(\xi), \quad \text{with } H(0) = 1, \tag{1.6}$$

where $p(\xi)$ is a trigonometric polynomial, $q(\xi) \in E_\mu$, and $q(\xi) \neq 0$, $\forall \xi \in \mathbb{R}^s$. The particular case $q(\xi) \equiv 1$ covers the case of finite supported masks.

Let M be a fixed integer matrix with $m = |\det M|$. Then the coset space $\mathbb{Z}^s / M^T \mathbb{Z}^s$ consists of m elements, where M^T denotes the transpose of M. Let $\omega_k + M^T \mathbb{Z}^s$, $k = 0, 1, \ldots, m-1$ be the m distinct elements of $\mathbb{Z}^s / M^T \mathbb{Z}^s$ with $\omega_0 = 0$. We denote $\Omega := \{\omega_k,\ k = 0, 1, \ldots, m-1\}$. Thus, each element $\alpha \in \mathbb{Z}^s$ can be uniquely represented as $\omega + M^T \varepsilon$, where $\omega \in \Omega$ and $\varepsilon \in \mathbb{Z}^s$. Analogously, we denote by Γ a complete set of representatives of the distinct cosets of $\mathbb{Z}^s / M \mathbb{Z}^s$.

We say that mask a satisfies the basic sum rule if

$$\sum_{\alpha \in \mathbb{Z}^s} a(\gamma + M\alpha) = \sum_{\alpha \in \mathbb{Z}^s} a(M\alpha) \quad \forall \gamma \in \Gamma.$$

For $1 \leqslant p \leqslant \infty$, denote by $\mathcal{L}_p(\mathbb{R}^s)$ the linear space of all (complex-valued) functions f such that $|f|_p < \infty$, where

$$|f|_p := \left(\int_{[0,1)^s} \left(\sum_{\alpha \in \mathbb{Z}^s} \left| f(\cdot - \alpha) \right| \right)^p dx \right)^{1/p} \quad \text{for } 1 \leqslant p < \infty,$$

and $|f|_\infty$ is the essential supremum of $\sum_{\alpha \in \mathbb{Z}^s} |f(\cdot - \alpha)|$ on $[0,1)^s$. Equipped with the norm $|\cdot|_p$, $\mathcal{L}_p(\mathbb{R}^s)$ becomes a Banach space. There are several important subspaces of $\mathcal{L}_p(\mathbb{R}^s)$. For instance, if $f \in L_p(\mathbb{R}^s)$ is compactly supported, then $f \in \mathcal{L}_p(\mathbb{R}^s)$ $(1 \leqslant p \leqslant \infty)$. Also, a function $f \in L_{p,\mu}(\mathbb{R}^s)$ with $\mu > 0$ is in $\mathcal{L}_p(\mathbb{R}^s)$ $(1 \leqslant p \leqslant \infty)$. See [19] for more discussions of $\mathcal{L}_p$ spaces.

The concept of stability plays an important role in the study of refinable functions. Let $\phi \in L_p(\mathbb{R}^s)$ $(1 \leqslant p \leqslant \infty)$. We say that the shifts of ϕ are ℓ_p-stable if there exist positive constants A_p and B_p such that for all sequences $a \in \ell_p(\mathbb{Z}^s)$,

$$A_p \|a\|_p \leqslant \left\| \sum_{\alpha \in \mathbb{Z}^s} a(\alpha) \phi(\cdot - \alpha) \right\|_p \leqslant B_p \|a\|_p. \tag{1.7}$$

See [19,22] for more details about ℓ_p-stability. It was proved by Jia and Micchelli in [19] that the shifts of a function $\phi \in \mathcal{L}_p(\mathbb{R}^s)$ are ℓ_p-stable if and only if, for any $\xi \in \mathbb{R}^s$, there exists an element $\beta \in \mathbb{Z}^s$ such that

$$\hat{\phi}(\xi + 2\pi\beta) \neq 0.$$

For $\nu \geqslant 0$, denote by $H^\nu(\mathbb{R}^s)$ the Sobolev space of all functions $f \in L_2(\mathbb{R}^s)$ such that

$$\int_{\mathbb{R}^s} \left| \hat{f}(\xi) \right|^2 \left(1 + |\xi|^2\right)^\nu d\xi < \infty.$$

The critical Sobolev exponent of a function $f \in L_2(\mathbb{R}^s)$ is defined by

$$s_f := \sup\left\{\nu:\ f \in H^\nu\left(\mathbb{R}^s\right)\right\}.$$

Sobolev spaces are closely related to Lipschitz spaces, which is defined in terms of the modulus of continuity. For $y \in \mathbb{R}^s$, the difference operator is defined by

$$\nabla_y f := f - f(\cdot - y),$$

where f is a function defined on $\mathbb{R}^s$. The modulus of continuity of a function f in $L_p(\mathbb{R}^s)$ is defined by

$$\omega(f,h)_p := \sup_{|y|\leqslant h} \|\nabla_y f\|_p, \quad h \geqslant 0.$$

Let k be a positive integer. The kth modulus of continuity of f in $L_p(\mathbb{R}^s)$ is defined by

$$\omega_k(f,h)_p := \sup_{|y|\leqslant h} \left\|\nabla_y^k f\right\|_p, \quad h \geqslant 0.$$

For $1 \leqslant p \leqslant \infty$, $0 < \nu \leqslant 1$ and a function $f \in L_p(\mathbb{R}^s)$, we say f belongs to the Lipschitz space $\mathrm{Lip}(\nu, L_p(\mathbb{R}^s))$ if there exists a positive constant C independent of h such that

$$w(f,h)_p \leqslant Ch^\nu \quad \forall h > 0.$$

For $\nu > 0$, we write $\nu = r + \eta$, where r is an integer and $0 < \eta \leqslant 1$. We say that f belongs to the Lipschitz space $\mathrm{Lip}(\nu, L_p(\mathbb{R}^s))$ if $D^\mu f \in \mathrm{Lip}(\eta, L_p(\mathbb{R}^s))$ for all multi-indices μ with $|\mu| = r$. For $\nu > 0$, let k be an integer greater than ν. The generalized Lipschitz space $\mathrm{Lip}^*(\nu, L_p(\mathbb{R}^s))$ consists of all functions $f \in L_p(\mathbb{R}^s)$ such that

$$\omega_k(f,h)_p \leqslant Ch^\nu \quad \forall h > 0,$$

where C is a positive constant independent of h. If $\nu > 0$ is not an integer, then

$$\mathrm{Lip}\big(\nu, L_p\big(\mathbb{R}^s\big)\big) = \mathrm{Lip}^*\big(\nu, L_p\big(\mathbb{R}^s\big)\big), \quad 1 \leqslant p \leqslant \infty.$$

See [8,35,17] for more details about Lipschitz spaces.

Following the discussion in [17], we have

$$\sup\big\{\nu\colon\ f \in H^\nu\big(\mathbb{R}^s\big)\big\} = \sup\big\{\nu\colon\ f \in \mathrm{Lip}\big(\nu, L_2\big(\mathbb{R}^s\big)\big)\big\} = \sup\big\{\nu\colon\ f \in \mathrm{Lip}^*\big(\nu, L_2\big(\mathbb{R}^s\big)\big)\big\}.$$

2. Some auxiliary results

In this section, we shall introduce some auxiliary results. Let $\mathcal{T} := [-\pi,\pi]^s$. We denote by $C(\mathcal{T})$ the space of all 2π-period continuous functions on $\mathbb{R}^s$ and by $L_\infty(\mathcal{T})$ the space of all 2π-period measurable functions f on $\mathcal{T}$ such that $\|f\|_{L_\infty(\mathcal{T})} < \infty$, where $\|f\|_{L_\infty(\mathcal{T})}$ denotes the essential supremum of $|f|$ on $\mathcal{T}$.

Following [6] and [28], it is easy to obtain Lemma 2.1 and Lemma 2.2.

Lemma 2.1. *Let a be an exponentially decaying sequence and $H(\xi)$ be given by (1.5) with $H(0) = 1$, then the infinite product*

$$\hat{\phi}(\xi) := \prod_{j=1}^{\infty} H\big(\big(M^T\big)^{-j}\xi\big) \tag{2.1}$$

converges absolutely and uniformly on any compact subsets of $\mathbb{R}^s$ to a continuous function.

Lemma 2.2. *For $\mu > 0$, let b be an exponentially decaying sequence with $\hat{b} \in E_\mu$ and $\hat{b}(0) = 1$, then for all positive integers N, there exist trigonometric polynomials*

$$\hat{b}_N(\xi) = \sum_{|\alpha|\leqslant N} b_N(\alpha) e^{-i\alpha\cdot\xi}, \quad \xi \in \mathbb{R}^s,$$

such that

$$\big\|\hat{b}(\xi) - \hat{b}_N(\xi)\big\|_{L_\infty(\mathcal{T})} \leqslant Ce^{-\mu N}, \quad \text{and} \quad \hat{b}_N(0) = 1,$$

where C is a positive constant independent of N and $|\alpha| = |\alpha_1| + \cdots + |\alpha_s|$ with $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{Z}^s$.

Proof. Let $\hat{b}_N(\xi)$ be defined by

$$\hat{b}_N(\xi) = \sum_{|\alpha|\leqslant N} b(\alpha) e^{-i\alpha\cdot\xi} + \sum_{|\alpha|>N} b(\alpha).$$

Then we have $\hat{b}_N(0) = 1$ and

$$\big\|\hat{b}(\xi) - \hat{b}_N(\xi)\big\|_\infty \leqslant \sum_{|\alpha|>N} 2\big|b(\alpha)\big| \leqslant Ce^{-\mu N},$$

where C is independent of N. $\square$

Transfer operator is a useful tool for the study of refinable functions (see e.g., [3,5,6,20,34]). For a given 2π-periodic function $u(\xi)$, the transition operator T_u acts on 2π-periodic functions according to

$$T_u f(\xi) = \sum_{\omega_i \in \Omega} u\big(\big(M^T\big)^{-1}(\xi + 2\pi\omega_i)\big) f\big(\big(M^T\big)^{-1}(\xi + 2\pi\omega_i)\big). \tag{2.2}$$

We fix $u(\xi) = |H(\xi)|^2$, where $H(\xi)$ is defined by (1.5) and $H(\xi) \in E_\mu$ for some $\mu > 0$. It is easy to see that $u(\xi) \in E_\mu$. In terms of the Fourier coefficients u_α of $u(\xi) = \sum_{\alpha \in \mathbb{Z}^s} u_\alpha e^{-i\alpha\cdot\xi}$, (2.2) can also be rewritten as

$$(T_u f)_\alpha = \frac{1}{(2\pi)^s} \int\limits_{[0,2\pi)^s} T_u f(\xi) e^{i\alpha\cdot\xi}\, d\xi = m \sum_{\beta \in \mathbb{Z}^s} u_{M\alpha-\beta} f_\beta, \tag{2.3}$$

where $f(\xi) = \sum_{\beta \in \mathbb{Z}^s} f_\beta e^{-i\beta\cdot\xi}$.

It follows from [14] that

$$\|T_u f\|_\mu = \sum_{\alpha \in \mathbb{Z}^s} \big|(T_u f)_\alpha\big| e^{\mu|\alpha|} = m \sum_{\alpha \in \mathbb{Z}^s} \left| \sum_{\beta \in \mathbb{Z}^s} u_{M\alpha-\beta} f_\beta \right| e^{\mu|\alpha|} \leqslant m\|u\|_\mu \|f\|_\mu.$$

Therefore, T_u is a bounded linear operator on E_μ. Furthermore, T_u is also a compact operator on E_μ.

Let $V = \{\hat{v} \in E_\mu\colon \sum_{\alpha \in \mathbb{Z}^s} v(\alpha) = 0\}$.

Theorem 2.3. *Let T_u be given by* (2.2) *with u being exponential decay. For $L \in \mathbb{N}$, define $z_L = [\sin^2(\frac{\xi_1}{2}) + \cdots + \sin^2(\frac{\xi_s}{2})]^L$. Then*

$$\varlimsup_{k\to\infty} \big\|T_u^k z_L\big\|_{L_\infty(\mathcal{T})}^{1/k} \leqslant \varlimsup_{k\to\infty} \big\|T_u^k z_L\big\|_\mu^{1/k} \leqslant \rho(T_u|_V), \tag{2.4}$$

where $\rho(T_u|_V)$ denotes the spectral radius of the restriction of T_u to the subspace V of E_μ.

Proof. It is easy to check that $z_L \in V$. Hence, for $k \geqslant 1$, we have

$$\big\|T_u^k z_L\big\|_{L_\infty(\mathcal{T})} \leqslant \big\|T_u^k z_L\big\|_\mu \leqslant \big\|T_u^k|_V\big\|_\mu \|z_L\|_\mu.$$

Since

$$\lim_{k\to\infty} \big\|T_u^k|_V\big\|_\mu^{1/k} = \rho(T_u|_V),$$

we obtain that

$$\varlimsup_{k\to\infty} \big\|T_u^k z_L\big\|_{L_\infty(\mathcal{T})}^{1/k} \leqslant \varlimsup_{k\to\infty} \big\|T_u^k z_L\big\|_\mu^{1/k} \leqslant \rho(T_u|_V). \qquad \square$$

Lemma 2.4. *(See [24,5].) Let T_u be given by* (2.2) *with u being exponential decay. For any $f(\xi), g(\xi) \in C(\mathcal{T})$, and any positive integer n, we have the following identity*

$$\int\limits_{\mathcal{T}} T_u^n f(\xi)\overline{g(\xi)}\, d\xi = \int\limits_{(M^T)^n\mathcal{T}} \left[\prod_{k=1}^n u\big(\big(M^T\big)^{-k}\xi\big)\right] f\big(\big(M^T\big)^{-n}\xi\big)\overline{g(\xi)}\, d\xi.$$

To characterize the regularity of refinable functions with masks exhibiting exponential decay, we need the following Theorem 2.5, which was established by Lorentz and Oswald in [28] for the special case $M = 2I_{s\times s}$.

Theorem 2.5. *Let $u(\xi) = |H(\xi)|^2 := P(\xi)Q(\xi)$, where $H(\xi)$ is defined by* (1.5) *and $P(\xi) = |p(\xi)|^2$, $Q(\xi) = |q(\xi)|^2$. Suppose $Q_N(\xi)$ is a trigonometric polynomial sequence of approximations of $Q(\xi)$ as in Lemma* 2.2 *such that $\|Q(\xi) - Q_N(\xi)\|_{L_\infty(\mathcal{T})} \leqslant Ce^{-\mu N}$. Let $u_N(\xi) = P(\xi)Q_N(\xi)$. Then*

$$\rho_L = \lim_{N\to\infty} \rho(T_{u_N}|_{V_{u_N,z_L}}) \tag{2.5}$$

and

$$\rho = \lim_{L\to\infty} \rho_L \tag{2.6}$$

both exist and do not depend on the specific choice of the sequence $Q_N(\xi)$, where V_{u_N,z_L} is defined by

$$V_{u_N,z_L} = \operatorname{span}\big\{T_{u_N}^n z_L\colon n \geqslant 0\big\}.$$

Proof. Since $Q(\xi)$ is a positive 2π-periodic continuous function, there exists a positive constant c such that $|Q(\xi)| \geqslant c > 0$. Consequently, for sufficiently large N, we have

$$\max\left(\left|1-\frac{Q_N(\xi)}{Q(\xi)}\right|, \left|1-\frac{Q(\xi)}{Q_N(\xi)}\right|\right) \leqslant C^* e^{-\mu N}, \tag{2.7}$$

where C^* is a positive constant independent of N.

Therefore,

$$\left(1-C^*e^{-\mu N}\right)Q_N(\xi) \leqslant Q(\xi) \leqslant \left(1+C^*e^{-\mu N}\right)Q_N(\xi),$$

and

$$\left(1-C^*e^{-\mu N}\right)u_N(\xi) \leqslant u(\xi) \leqslant \left(1+C^*e^{-\mu N}\right)u_N(\xi).$$

By Lemma 2.4, we have

$$\left(1-C^*e^{-\mu N}\right)\left\|T_{u_N}^k z_L\right\|_{L_\infty(\mathcal{T})}^{1/k} \leqslant \left\|T_u^k z_L\right\|_{L_\infty(\mathcal{T})}^{1/k} \leqslant \left(1+C^*e^{-\mu N}\right)\left\|T_{u_N}^k z_L\right\|_{L_\infty(\mathcal{T})}^{1/k}. \tag{2.8}$$

Since u_N is a trigonometric polynomial, by [13, Lemma 2.4], we have that V_{u_N,Z_L} is finite dimensional and

$$\lim_{k\to\infty}\left\|T_{u_N}^k z_L\right\|_{L_\infty(\mathcal{T})}^{1/k} = \rho(T_{u_N}|_{V_{u_N,Z_L}}).$$

Letting $k \to \infty$ in (2.8), by Theorem 2.3, we have that

$$\left(1-C^*e^{-\mu N}\right)\rho(T_{u_N}|_{V_{u_N,Z_L}}) \leqslant \overline{\lim_{k\to\infty}}\left\|T_u^k z_L\right\|_{L_\infty(\mathcal{T})}^{1/k} \leqslant \left(1+C^*e^{-\mu N}\right)\rho(T_{u_N}|_{V_{u_N,Z_L}}). \tag{2.9}$$

Then, letting $N \to \infty$ in (2.9), we obtain that

$$\rho_L := \lim_{N\to\infty}\rho(T_{u_N}|_{V_{u_N,Z_L}}) = \overline{\lim_{k\to\infty}}\left\|T_u^k z_L\right\|_{L_\infty(\mathcal{T})}^{1/k} \tag{2.10}$$

exists.

If $\tilde{u}_N(\xi)$ is another sequence of approximations of $u(\xi)$, we can easily obtain

$$\left(1-C^*e^{-\mu N}\right)u_N(\xi) \leqslant \tilde{u}_N(\xi) \leqslant \left(1+C^*e^{-\mu N}\right)u_N(\xi).$$

Thus,

$$\left(1-C^*e^{-\mu N}\right)^k\left\|T_{u_N}^k z_L\right\|_{L_\infty(\mathcal{T})} \leqslant \left\|T_{\tilde{u}_N}^k z_L\right\|_{L_\infty(\mathcal{T})} \leqslant \left(1+C^*e^{-\mu N}\right)^k\left\|T_{u_N}^k z_L\right\|_{L_\infty(\mathcal{T})}.$$

This together with (2.10) implies that ρ_L is independent of the specific choice of the sequence u_N. Since $0 \leqslant z_{L+1} \leqslant sz_L$, (2.10) yields

$$\rho = \lim_{L\to\infty}\rho_L \leqslant \rho_{L+1} \leqslant \rho_L \leqslant \cdots \leqslant \rho_0. \tag{2.11}$$

The proof of Theorem 2.5 is complete. □

3. Characterization of regularity

In this section, we shall give some characterizations of the regularity of refinable functions with mask a exhibiting exponential decay and a dilation matrix M whose eigenvalues have the same modulus.

When mask a is finitely supported, $H(\xi) = \frac{1}{m}\sum_{\alpha\in\mathbb{Z}^s} a(\alpha)e^{-i\alpha\cdot\xi}$ is a trigonometric polynomial. Define S_c as the largest nonnegative integer such that

$$D^\mu H\left(2\pi\left(M^T\right)^{-1}\omega\right) = 0 \quad \forall\omega \in \Omega\backslash\{0\}, \quad \text{and} \quad |\mu| \leqslant S_c - 1.$$

In the following, L denotes an integer larger than S_c.

It follows from [5] that $V_{|H|^2,Z_L} = \operatorname{span}\{T_{|H|^2}^k z_L,\ k \geqslant 0\}$ is finite dimensional and is an invariant subspace of $T_{|H|^2}$.

Cohen, Gröchenig, and Villemoes [5] gave a characterization of the regularity of refinable functions based on the spectral radius of $T_{|H|^2}$ restricted to $V_{|H|^2,Z_L}$ when mask a is finitely supported. Their results were stated as follows:

Theorem 3.1 *(Cohen et al.). Let M be a dilation matrix whose eigenvalues have the same modulus and $m = |\det M|$. Let $\phi \in L_2(\mathbb{R}^s)$ be a compactly supported solution to* (1.1) *with mask a being finitely supported. Suppose the shifts of ϕ are stable. Let ρ be the spectral radius of $T_{|H|^2}$ restricted to $V_{|H|^2,Z_L}$. Then the critical Sobolev exponent*

$$s_\phi = -\frac{s}{2}\log_m \rho, \tag{3.1}$$

and $\phi \in H^s$ if and only if $s < s_\phi$.

We point out that similar characterization was also established by Jia in [17] with a different method. Several other researchers have considered the regularity of refinable functions in high dimensions (see [4,21,23–26,30,33,34]).

We are now in a position to establish the following characterizations of regularity of refinable functions with exponentially decaying masks and a dilation matrix whose eigenvalues have the same modulus.

Theorem 3.2. *Let M be a dilation matrix whose eigenvalues have the same modulus and $m = |\det M|$. Let ϕ be an L_2-solution of refinement equation* (1.1) *with mask a being exponential decay. Assume that $H(\xi)$ satisfies* (1.6)*. Let ρ be defined by* (2.6)*. Then $s_\phi \geqslant -\frac{s}{2}\log_m \rho$. Moreover, if the shifts of ϕ are stable, then $s_\phi = -\frac{s}{2}\log_m \rho$.*

Proof. Set $\mathcal{T} = [-\pi,\pi]^s$, $G = \mathcal{T}\backslash(M^T)^{-1}\mathcal{T}$, then $\mathbb{R}^s = \mathcal{T} \cup \bigcup_{n=1}^{\infty}(M^T)^n G$. For $\xi \in (M^T)^n\mathcal{T}$, it is easy to see that

$$\left|\hat{\phi}(\xi)\right|^2 \leqslant C_1 \prod_{k=1}^{n}\left|H\left(\left(M^T\right)^{-k}\xi\right)\right|^2, \tag{3.2}$$

with $C_1 = \max_{\xi\in\mathcal{T}}|\hat{\phi}(\xi)|^2$. For any fixed $L \in \mathbb{N}$, since $(M^T)^{-1}\mathcal{T}$ contains an open ball centered at the origin, we have

$$z_L(\xi) \geqslant C_2 > 0 \quad \text{for } \xi \in G. \tag{3.3}$$

It follows from Lemma 2.4,

$$\int\limits_{(M^T)^n G}\left[\prod_{k=1}^{n}\left|H\left(\left(M^T\right)^{-k}\xi\right)\right|^2\right] z_L\left(\left(M^T\right)^{-n}\xi\right)d\xi = \left\langle T^n_{|H|^2}z_L, 1\right\rangle \leqslant \left\|T^n_{|H|^2}z_L\right\|_{L_\infty(\mathcal{T})}. \tag{3.4}$$

Thus, combining (3.2), (3.3) and (3.4), we have

$$\int\limits_{(M^T)^n G}\left|\hat{\phi}(\xi)\right|^2\left(1+|\xi|^2\right)^\gamma d\xi \leqslant C_3 m^{\frac{2\gamma n}{s}}\int\limits_{(M^T)^n G}\left|\hat{\phi}(\xi)\right|^2 d\xi \leqslant C_4 m^{\frac{2\gamma n}{s}}\left\|T^n_{|H|^2}z_L\right\|_{L_\infty(\mathcal{T})}. \tag{3.5}$$

Note that,

$$\int\limits_{\mathbb{R}^s}\left|\hat{\phi}(\xi)\right|^2\left(1+|\xi|^2\right)^\gamma d\xi = \int\limits_{\mathcal{T}}\left|\hat{\phi}(\xi)\right|^2\left(1+|\xi|^2\right)^\gamma d\xi + \sum_{n=1}^{\infty}\int\limits_{(M^T)^n G}\left|\hat{\phi}(\xi)\right|^2\left(1+|\xi|^2\right)^\gamma d\xi. \tag{3.6}$$

Since $|\hat{\phi}(\xi)|^2(1+|\xi|^2)^\gamma$ is continuous on $\mathcal{T}$, the first integer on the right-hand side of (3.6) is a finite constant C_5. This together with (3.5) implies that

$$\int\limits_{\mathbb{R}^s}\left|\hat{\phi}(\xi)\right|^2\left(1+|\xi|^2\right)^\gamma d\xi \leqslant C_5 + C_4\sum_{n=1}^{\infty}m^{\frac{2\gamma n}{s}}\left\|T^n_{|H|^2}z_L\right\|_{L_\infty(\mathcal{T})}.$$

By Theorem 2.5, for any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that for large enough L,

$$\left\|T^n_{|H|^2}z_L\right\|_{L_\infty(\mathcal{T})} \leqslant C(\varepsilon)(\rho+\varepsilon)^n, \quad n \in \mathbb{N}.$$

Consequently,

$$\int\limits_{\mathbb{R}^s}\left|\hat{\phi}(\xi)\right|^2\left(1+|\xi|^2\right)^\gamma d\xi \leqslant C_5 + C_4 C(\varepsilon)\sum_{n=1}^{\infty}m^{\frac{2\gamma n}{s}}(\rho+\varepsilon)^n.$$

When $\gamma < -\frac{s}{2}\log_m(\rho+\varepsilon)$,

$$\int\limits_{\mathbb{R}^s}\left|\hat{\phi}(\xi)\right|^2\left(1+|\xi|^2\right)^\gamma d\xi < \infty.$$

Since ε can be chosen arbitrary small, we conclude that $s_\phi \geqslant -\frac{s}{2}\log_m \rho$ and the first part of the theorem follows. In all these estimates, the positive constants C_j are independent of n.

To prove the second part, for $H(\xi) = p(\xi)q(\xi)$, we can choose a sequence of trigonometric polynomials $q_N(\xi)$ of approximations of $q(\xi)$ as in Lemma 2.2. Since the shifts of ϕ are stable, it follows from [14, Theorem 4.1] that the cascade algorithm associated with mask a and dilation matrix M converges in $L_2(\mathbb{R}^s)$. Therefore, a satisfies the basic sum rule and $\rho(T_{|H|^2}|_V) < 1$. Let

$$H_N(\xi) = p(\xi)q_N(\xi),$$

then $H_N(\xi)$ generate a sequence of refinable functions $\phi_N(\xi)$ with compact support. Since a satisfies the basic sum rule if and only if $H(2\pi(M^T)^{-1}\omega) = 0$, $\forall\omega \in \Omega\backslash\{0\}$. It follows that the masks a_N associated with ϕ_N also satisfy the basic sum rule for N large enough. By the choice of H_N, we have that

$$\lim_{N\to\infty} T_{|H_N|^2} = T_{|H|^2}$$

in the E_μ-norm, which implies that

$$\rho(T_{|H_N|^2}|_V) < 1,$$

for N large enough. Then, it follows from [14, Theorem 4.2] that the cascade algorithms associated with masks a_N and the dilation matrix M converge in $L_2(\mathbb{R}^s)$, which implies that $\phi_N \in L_2(\mathbb{R}^s)$.

Since

$$\hat{\phi}(\xi) = \prod_{j=1}^{\infty} H\left(\frac{\xi}{2^j}\right) = \prod_{j=1}^{\infty} p\left(\frac{\xi}{2^j}\right)\prod_{j=1}^{\infty} q\left(\frac{\xi}{2^j}\right),$$

and

$$\hat{\phi}_N(\xi) = \prod_{j=1}^{\infty} H_N\left(\frac{\xi}{2^j}\right) = \prod_{j=1}^{\infty} p\left(\frac{\xi}{2^j}\right)\prod_{j=1}^{\infty} q_N\left(\frac{\xi}{2^j}\right).$$

If the shifts of ϕ are stable, we obtain that there exist positive constants A and B such that

$$A \leqslant \sum_{k\in\mathbb{Z}^s}\left|\hat{\phi}(\xi + 2\pi k)\right|^2 \leqslant B \quad \forall\xi \in \mathbb{R}^s.$$

Thus, for any $\xi \in \mathbb{R}^s$, there exists an $\alpha \in \mathbb{Z}^s$ such that $\hat{\phi}(\xi + 2\pi\alpha) \neq 0$, which implies that $\prod_{j=1}^{\infty} p(\frac{\xi+2\pi\alpha}{2^j}) \neq 0$. Therefore, $\hat{\phi}_N(\xi + 2\pi\alpha) \neq 0$ for N large enough. We obtain that the shifts of ϕ_N are also stable for sufficiently large N.

Assume $\phi \in H^\gamma(\mathbb{R}^s)$ for some $\gamma > -\frac{s}{2}\log_m \rho$, then

$$\int_{\mathbb{R}^s} \left|\hat{\phi}(\xi)\right|^2 \left(1 + |\xi|^2\right)^\gamma d\xi < \infty.$$

Fix any $\tilde{\gamma} \in (-\frac{s}{2}\log_m \rho, \gamma)$, we claim that $\phi_N \in H^{\tilde{\gamma}}(\mathbb{R}^s)$ for N large enough.

For sufficiently large N, we have $1 + C^* e^{-\mu N} < m^{\frac{2(\gamma-\tilde{\gamma})}{s}}$, where C^* is the same as in (2.7). Set $\Omega_k = (M^T)^k\mathcal{T}\backslash(M^T)^{(k-1)}\mathcal{T}$, it is easy to see that $\mathbb{R}^s = (\bigcup_{k=1}^{\infty}\Omega_k) \cup \mathcal{T}$. It follows that

$$\begin{aligned}
&\int_{\mathbb{R}^s} \left(1 + |\xi|^2\right)^{\tilde{\gamma}} \left|\hat{\phi}_N(\xi)\right|^2 d\xi \\
&\quad = \int_{\mathcal{T}} \sum_{\alpha\in\mathbb{Z}^s} \left(1 + |\xi + 2\pi\alpha|^2\right)^{\tilde{\gamma}} \left(\prod_{k=1}^{\infty}\left|H_N\left(\left(M^T\right)^{-k}(\xi + 2\pi\alpha)\right)\right|^2\right) d\xi \\
&\quad \leqslant C\sum_{k=1}^{\infty} m^{\frac{2\tilde{\gamma}k}{s}} \int_{\mathcal{T}} \sum_{\alpha\in\Omega_k} \left(\prod_{l=1}^{k}\left|H_N\left(\left(M^T\right)^{-l}(\xi + 2\pi\alpha)\right)\right|^2\right) d\xi + \int_{\mathcal{T}} \sum_{\alpha\in\mathcal{T}} \left(1 + |\xi + 2\pi\alpha|^2\right)^{\tilde{\gamma}} \left|\hat{\phi}_N(\xi + 2\pi\alpha)\right|^2 d\xi.
\end{aligned}$$

Note that $|\hat{\phi}_N(\xi + 2\pi\alpha)|^2$ is continuous on $\mathcal{T}$. Hence, there exists a positive constant B_1 such that $\int_{\mathcal{T}} \sum_{\alpha\in\mathcal{T}}(1 + |\xi + 2\pi\alpha|^2)^{\tilde{\gamma}}|\hat{\phi}_N(\xi + 2\pi\alpha)|^2 d\xi \leqslant B_1$.

Consequently, we deduce that

$$\begin{aligned}
&\int_{\mathbb{R}^s} \left(1 + |\xi|^2\right)^{\tilde{\gamma}} \left|\hat{\phi}_N(\xi)\right|^2 d\xi \\
&\quad \leqslant C\sum_{k=1}^{\infty} m^{\frac{2\gamma k}{s}} \left(1 + C^* e^{-\mu N}\right)^{-k} \int_{\mathcal{T}} \sum_{\alpha\in\Omega_k} \left(\prod_{l=1}^{k}\left|H_N\left(\left(M^T\right)^{-l}(\xi + 2\pi\alpha)\right)\right|^2\right) d\xi + B_1
\end{aligned}$$

$$\leqslant C\sum_{k=1}^{\infty} m^{\frac{2\gamma k}{s}} \int_{\mathcal{T}} \sum_{\alpha\in\Omega_k}\left[\prod_{l=1}^{k}\left|H\left(\left(M^T\right)^{-l}(\xi+2\pi\alpha)\right)\right|^2\right]d\xi + B_1$$

$$= C\int_{\mathcal{T}} \sum_{k=1}^{\infty} m^{\frac{2\gamma k}{s}} \sum_{\alpha\in\Omega_k}\left[\prod_{l=1}^{k}\left|H\left(\left(M^T\right)^{-l}(\xi+2\pi\alpha)\right)\right|^2\right]d\xi + B_1$$

$$\leqslant \tilde{C}\int_{\mathbb{R}^s} \left|\hat{\phi}(\xi)\right|^2\left(1+|\xi|^2\right)^{\gamma} d\xi + B_1 < \infty.$$

Therefore, $\phi_N \in H^{\tilde{\gamma}}(\mathbb{R}^s)$ for N large enough. Since the shifts of ϕ_N are stable, it follows from Theorem 3.1 that $s_{\phi_N} = -\frac{s}{2}\log_m \rho(T_{u_N}, V_{u_N,Z_L})$ for L large enough. Therefore, $s_{\phi_N} \geqslant \tilde{\gamma} > -\frac{s}{2}\log_m \rho$ for N large enough. By the definition of ρ, this is impossible. Hence, $s_\phi \leqslant -\frac{s}{2}\log_m \rho$. Combined with the proof of the first part, we complete the proof of Theorem 3.2. □

Remark 3.3. Theorem 3.2 characterizes the optimal smoothness of a refinable function with mask a having exponential decay and a dilation matrix M whose eigenvalues have the same modulus, which extends Theorem 3.1 to the case that mask a is infinitely supported. Theorem 3.2 was also established in [28] for the case $M = 2I_{s\times s}$.

In some cases, when mask a exhibits exponential decay, the solution of Eq. (1.1) belongs to $\mathcal{L}_p(\mathbb{R}^s)$ $(1 \leqslant p \leqslant \infty)$. For example, Han [12] characterized the existence of $L_{2,\mu}$-solution of Eq. (1.1) with a being exponential decay and $s = 1$, $M = 2$. Jia [18] also gave a characterization of the existence of the solution of refinement equation in $L_{p,\mu}$ $(1 \leqslant p \leqslant \infty)$ by considering the convergence of the cascade algorithm associated with an exponentially decaying mask. Hogan [16] investigated some properties of refinement equation (1.1) under the assumptions that the solution $\phi \in \mathcal{L}_p(\mathbb{R}^s)$ for $1 \leqslant p \leqslant \infty$. The following Theorems 3.4 and 3.5 will give some characterizations of the regularity of refinable functions in $\mathcal{L}_2(\mathbb{R}^s)$ and in $\mathcal{L}_p(\mathbb{R}^s)$ $(1 \leqslant p \leqslant \infty)$, respectively. Our characterizations are based on a discrete version of Young's inequality.

Theorem 3.4. *Suppose $\nu > 0$ and k is a positive integer. Let M be a dilation matrix whose eigenvalues have the same modulus and $m = |\det M|$. Suppose $\phi \in \mathcal{L}_2(\mathbb{R}^s)$ is the normalized solution of* (1.1) *with mask a being exponential decay. For $n = 1, 2, \ldots$, let a_n be given by*

$$a_n(\alpha) = \sum_{\beta\in\mathbb{Z}^s} a(\alpha - M\beta)a_{n-1}(\beta), \quad \alpha \in \mathbb{Z}^s, \tag{3.7}$$

where a_0 is the δ sequence given by $\delta(0) = 1$ and $\delta(\alpha) = 0$ for $\alpha \in \mathbb{Z}^s\backslash\{0\}$.

If there exists a constant $C > 0$ such that

$$\left\|\nabla_j^k a_n\right\|_2 \leqslant Cm^{(1/2-\nu/s)n} \quad \forall n \in \mathbb{N} \text{ and } j = 1, \ldots, s, \tag{3.8}$$

then $\phi \in \mathrm{Lip}^*(\nu, L_2(\mathbb{R}^s))$. Conversely, if $\phi \in \mathrm{Lip}^*(\nu, L_2(\mathbb{R}^s))$ and the shifts of ϕ are stable, then (3.8) *holds for $k > \nu$.*

Proof. Since $\phi \in \mathcal{L}_2(\mathbb{R}^s)$ is the normalized solution of (1.1), it is easy to check that

$$\phi = \sum_{\alpha\in\mathbb{Z}^s} a_n(\alpha)\phi\left(M^n \cdot -\alpha\right). \tag{3.9}$$

Therefore,

$$\nabla_{M^{-n}e_j}^k \phi = \sum_{\alpha\in\mathbb{Z}^s} \nabla_j^k a_n(\alpha)\phi\left(M^n \cdot -\alpha\right). \tag{3.10}$$

By (1.2), we obtain that

$$\left\|\nabla_{M^{-n}e_j}^k \phi\right\|_2 = \left(\int_{\mathbb{R}^s}\left|\sum_{\alpha\in\mathbb{Z}^s} \nabla_j^k a_n \phi\left(M^n x - \alpha\right)\right|^2 dx\right)^{1/2}$$

$$= m^{-\frac{n}{2}}\left(\sum_{\beta\in\mathbb{Z}^s}\int_{\beta+[0,1]^s}\left|\sum_{\alpha\in\mathbb{Z}^s} \nabla_j^k a_n(\alpha)\phi(x-\alpha)\right|^2 dx\right)^{1/2}$$

$$= m^{-\frac{n}{2}}\left(\sum_{\beta\in\mathbb{Z}^s}\int_{[0,1]^s}\left|\sum_{\alpha\in\mathbb{Z}^s} \nabla_j^k a_n(\alpha)\phi(x-\beta-\alpha)\right|^2 dx\right)^{1/2}$$

$$= m^{-\frac{n}{2}} \Bigg(\int\limits_{[0,1]^s} \sum_{\beta \in \mathbb{Z}^s} \Bigg| \sum_{\alpha \in \mathbb{Z}^s} \nabla_j^k a_n(\alpha) \phi(x-\beta-\alpha) \Bigg|^2 dx \Bigg)^{1/2}$$

$$\leqslant m^{-\frac{n}{2}} \Bigg(\int\limits_{[0,1]^s} \|\nabla_j^k a_n\|_2^2 \Big(\sum_{\beta \in \mathbb{Z}^s} |\phi(x-\beta)| \Big)^2 dx \Bigg)^{1/2}$$

$$\leqslant m^{-\frac{n}{2}} \|\nabla_j^k a_n\|_2 |\phi|_{\mathcal{L}_2[0,1]^s}.$$

This in connection with (3.8) tells us that

$$\|\nabla_{M^{-n}e_j}^k \phi\|_2 \leqslant C m^{-\frac{\nu n}{s}} |\phi|_{\mathcal{L}_2[0,1]^s}. \tag{3.11}$$

Similar to [11,10], since all the eigenvalues of M have the same modulus, for any $\rho_1 > 1$, there exists a norm $|\cdot|$ on $\mathbb{R}^s$ such that the inequalities

$$C_1 \big(\rho_1^{-1} m^{-\frac{1}{s}}\big)^n |v| \leqslant |M^{-n} v| \leqslant C_2 \big(\rho_1 m^{-\frac{1}{s}}\big)^n |v|$$

hold true for every $v \in \mathbb{R}^s$, where C_1 and C_2 are positive constants independent of n.

Let y be any nonzero vector in $\mathbb{R}^s$. Then there exists positive integer n such that

$$1 \leqslant |M^n y| \leqslant C_2 \big(\rho_1 m^{\frac{1}{s}}\big)^n |y|. \tag{3.12}$$

By using the same argument as in [17, Theorem 2.1], from (3.11) and (3.12), we conclude that

$$\|\nabla_y^k \phi\|_2 \leqslant C_3 \big({\rho_1}^{\nu n} |y|^\nu\big).$$

Letting $\rho_1 \to 1$, we have

$$\|\nabla_y^k \phi\|_2 \leqslant C_3 |y|^\nu \quad \forall y \in \mathbb{R}^s. \tag{3.13}$$

This shows $\phi \in \mathrm{Lip}^*(\nu, L_2(\mathbb{R}^s))$, as desired.

Conversely, since the shifts of ϕ are stable. It follows from (1.7) and (3.10) that

$$m^{-\frac{n}{2}} \|\nabla_j^k a_n\|_2 \leqslant C_4 \|\nabla_{M^{-n}e_j}^k \phi\|_2, \tag{3.14}$$

where C_4 is a constant independent of n and j. If $\phi \in \mathrm{Lip}^*(\nu, L_2(\mathbb{R}^s))$, then for $k > \nu$,

$$\|\nabla_{M^{-n}e_j}^k \phi\|_2 \leqslant C |M^{-n} e_j|^\nu \leqslant C C_2 \big(\rho_1 m^{-\frac{1}{s}}\big)^{\nu n}.$$

Letting $\rho_1 \to 1$, we have

$$\|\nabla_{M^{-n}e_j}^k \phi\|_2 \leqslant C C_2 m^{-\frac{\nu n}{s}}. \tag{3.15}$$

Therefore, (3.8) follows from (3.14) and (3.15) immediately. □

Theorem 3.5. *Let $\nu > 0$ and k be a positive integer. Suppose $M = qI_{s\times s}$ and $q \geqslant 2$ is an integer. Let $\phi \in \mathcal{L}_p(\mathbb{R}^s)$ $(1 \leqslant p \leqslant \infty)$ be the normalized solution of* (1.1) *with mask a being exponential decay. For $n = 1, 2, \ldots,$ let a_n be given by* (3.7)*. If there exists a constant $C > 0$ such that*

$$\|\nabla_j^k a_n\|_p \leqslant C q^{(s/p-\nu)n} \quad \forall n \in \mathbb{N} \text{ and } j = 1, \ldots, s, \tag{3.16}$$

then $\phi \in \mathrm{Lip}^*(\nu, L_p(\mathbb{R}^s))$. Conversely, if $\phi \in \mathrm{Lip}^*(\nu, L_p(\mathbb{R}^s))$ and the shifts of ϕ are ℓ_p-stable, then (3.16) *holds for $k > \nu$.*

Proof. Since ϕ is the normalized solution of (1.1), we have that

$$\nabla_{q^{-n}e_j}^k \phi = \sum_{\alpha \in \mathbb{Z}^s} \nabla_j^k a_n(\alpha) \phi\big(q^n \cdot - \alpha\big). \tag{3.17}$$

By virtue of (1.2), we have

$$\begin{aligned}
\left\|\nabla^k_{q^{-n}e_j}\phi\right\|_p &= \left(\int_{\mathbb{R}^s}\left|\sum_{\alpha\in\mathbb{Z}^s}\nabla^k_j a_n\phi\left(q^n x-\alpha\right)\right|^p dx\right)^{1/p} \\
&= q^{-\frac{ns}{p}}\left(\sum_{\beta\in\mathbb{Z}^s}\int_{\beta+[0,1]^s}\left|\sum_{\alpha\in\mathbb{Z}^s}\nabla^k_j a_n(\alpha)\phi(x-\alpha)\right|^p dx\right)^{1/p} \\
&= q^{-\frac{ns}{p}}\left(\sum_{\beta\in\mathbb{Z}^s}\int_{[0,1]^s}\left|\sum_{\alpha\in\mathbb{Z}^s}\nabla^k_j a_n(\alpha)\phi(x-\beta-\alpha)\right|^p dx\right)^{1/p} \\
&= q^{-\frac{ns}{p}}\left(\int_{[0,1]^s}\sum_{\beta\in\mathbb{Z}^s}\left|\sum_{\alpha\in\mathbb{Z}^s}\nabla^k_j a_n(\alpha)\phi(x-\beta-\alpha)\right|^p dx\right)^{1/p} \\
&\leqslant q^{-\frac{ns}{p}}\left(\int_{[0,1]^s}\left\|\nabla^k_j a_n\right\|_p^p\left(\sum_{\beta\in\mathbb{Z}^s}\left|\phi(x-\beta)\right|\right)^p dx\right)^{1/p} \\
&\leqslant q^{-\frac{ns}{p}}\left\|\nabla^k_j a_n\right\|_p |\phi|_{\mathcal{L}_p[0,1]^s}.
\end{aligned}$$

By (3.16), we obtain that

$$\left\|\nabla^k_{q^{-n}e_j}\phi\right\|_p \leqslant Cq^{-\nu n}|\phi|_{\mathcal{L}_p[0,1]^s}. \tag{3.18}$$

From [26, Theorem 3.1], we know that (3.18) is equivalent to

$$\left\|\nabla^k_y\phi\right\|_p \leqslant C|y|^\nu \quad \forall y\in\mathbb{R}^s. \tag{3.19}$$

This shows $\phi\in\mathrm{Lip}^*(\nu, L_p(\mathbb{R}^s))$, as desired.

Conversely, since the shifts of ϕ are ℓ_p-stable, it follows from (1.7) and (3.17) that

$$q^{-\frac{ns}{p}}\left\|\nabla^k_j a_n\right\|_p \leqslant C_2\left\|\nabla^k_{q^{-n}e_j}\phi\right\|_p, \tag{3.20}$$

where C_2 is a constant independent of n and j. If $\phi\in\mathrm{Lip}^*(\nu, L_p(\mathbb{R}^s))$, then for $k>\nu$

$$\left\|\nabla^k_{q^{-n}e_j}\phi\right\|_p \leqslant Cq^{-\nu n}. \tag{3.21}$$

Therefore, (3.16) follows from (3.20) and (3.21). The proof of the theorem is complete. □

Remark 3.6. We remark that Theorem 3.4 was established by Jia in [17] for the case in which ϕ is a compactly supported L_2-solution of Eq. (1.1) and Theorem 3.5 was obtained by Li in [26] and by Jia et al. in [21] for the case in which ϕ is a compactly supported L_p-solution of Eq. (1.1) for $1\leqslant p\leqslant\infty$.

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